ON BRAIDED POISSON AND QUANTUM INHOMOGENEOUS GROUPS

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The well known incompatibility between inhomogeneous quantum groups and the standard q-deformation is shown to disappear (at least in certain cases) when admitting the quantum group to be braided. Braided quantum $ISO(p, N{-}p)$ containing $SO_q(p, N{-}p)$ with $|q| = 1$ are constructed for $N = 2p$, $2p + 1$, $2p + 2$. Their Poisson analogues (obtained first) are presented as an introduction to the quantum case.

1 Introduction

It is well known [1, 2] that the Lorentz part of any quantum (or Poisson) Poincaré group is triangular. This is in fact a general feature, which excludes the standard q-deformation from the context of inhomogeneous quantum groups [3]. In order to make the standard q-deformation compatible with inhomogeneous groups one has to consider some generalization of the notion of quantum (Poisson) group, such as, for example, a braided quantum (Poisson) group.

The notion of a braided Hopf algebra is due to S. Majid [4]. It is a natural generalization of the notion of a Hopf algebra when we replace the usual symmetric monoidal category of vector spaces by a braided one (the incorporation of *-structures is more controversial — we follow here the approach of [5]). A characteristic feature of this generalization is that the comultiplication is a morphism of algebras when the product algebra is considered with a crossed tensor product structure rather than the ordinary one.

On the Poisson level, it means that instead of ordinary Poisson groups (G, π) (where π is such a Poisson structure on G that the group multiplication is a Poisson map from the usual product Poisson structure $\pi \oplus \pi$ on $G \times G$ to π on G), we consider triples $(G, \pi, \pi_{\bowtie})$, where π is a Poisson structure on G and $\pi_{\bowtie}$ is a bi-vector field on $G \times G$ of the cross-type (i.e. having zero both projections on G) such that

1. $\pi_{12} := \pi \oplus \pi + \pi_{\bowtie}$ is a Poisson structure on $G \times G$,

2. the group multiplication is a Poisson map from π_{12} to π.

In the next section we shall construct such structures on the inhomogeneous orthogonal groups $ISO(p, p)$, $ISO(p, p{+}1)$, $ISO(p, p{+}2)$, with the homogeneous part being non-triangular (with standard Belavin-Drinfeld r-matrix).

In Sect. 3, similar result is obtained for the quantum case.

2 The Poisson case

In this section we discuss Poisson-Lie structures (possibly braided) on inhomogeneous orthogonal groups (in particular, on the Poincaré group). Let $V \cong \mathbb{R}^N = \mathbb{R}^{p+(N-p)}$ be equipped with the standard scalar product η of signature $(p, N-p)$. Special linear transformations preserving η form the *homogeneous* orthogonal group $H := SO(p, N-p) \subset GL(V)$ with the Lie algebra $\mathfrak{h} := so(p, N-p) \subset \operatorname{End} V$. The corresponding *inhomogeneous* group $G = V \rtimes H$ (with Lie algebra $\mathfrak{g} = V \rtimes \mathfrak{h}$) may be identified with the set of matrices

$$G = \left\{ \left(\begin{array}{c|c} h & x \\ \hline 0 & 1 \end{array} \right) \in \operatorname{End}(V \oplus \mathbb{R}) : h \in H,\, x \in V \right\}. \tag{1}$$

For $N > 2$, any multiplicative bi-vector field π on G is known [2] to be of the form $\pi(g) = \pi_r(g) := gr - rg$, where $r \in \bigwedge^2 \mathfrak{g}$. Here r has three components,

$$r = a + b + c \ \in \ (\bigwedge^2 V) \oplus (V \wedge \mathfrak{h}) \ \oplus (\bigwedge^2 \mathfrak{h}). \tag{2}$$

Decomposing $(V \oplus \mathbb{R}) \otimes (V \oplus \mathbb{R}) = (V \otimes V) \oplus (V \otimes \mathbb{R}) \oplus (\mathbb{R} \otimes V) \oplus (\mathbb{R} \otimes \mathbb{R})$ (in this order), we can write tensor product of matrices again as matrices:

$$g_1 g_2 = \begin{pmatrix} h_1 h_2 & h_1 x_2 & x_1 h_2 & x_1 x_2 \\ 0 & h_1 & 0 & x_1 \\ 0 & 0 & h_2 & x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} c & -b_{21} & b & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3}$$

where the subscripts 1,2 denote the insertion place in the tensor product. Using this, we obtain more detailed description of the brackets defined by π,

$$\{g_1, g_2\} = r g_1 g_2 - g_1 g_2 r, \tag{4}$$

as $\{h_1, h_2\} = c h_1 h_2 - h_1 h_2 c$, $\{x_1, h_2\} = c x_1 h_2 + b h_2 - h_1 h_2 b_{21}$ and $\{x_1, x_2\} = c x_1 x_2 + b x_2 - b_{21} x_1 + a - h_1 h_2 a$. It follows that with any Poisson group structure on G there is associated a Poisson group structure on H (with c being the r-matrix) and the projection from G to H is a Poisson map. As shown in [2] (see also below), c must be triangular (hence non-standard). The problem now arises if a non-triangular c can be used to construct (at least) a braided Poisson G.

Let us simplify the discussion to the case when $r = c$ (note that then the inclusion $H \subset G$ is also a Poisson map). The brackets have now the form

$$\{h_1, h_2\} = r h_1 h_2 - h_1 h_2 r, \qquad \{x_1, h_2\} = r x_1 h_2, \qquad \{x_1, x_2\} = r x_1 x_2. \tag{5}$$

We shall show that these brackets are not Poisson, unless r is triangular. It is convenient to check if the Jacobi identity is satisfied in a slightly more general case:

$$\{h_1, h_2\} = r h_1 h_2 - h_1 h_2 r, \qquad \{x_1, h_2\} = w x_1 h_2, \qquad \{x_1, x_2\} = r x_1 x_2, \tag{6}$$

where $w \in \mathfrak{h} \otimes \mathfrak{h}$. Let $J(f_1, f_2, f_3) := \{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\}$ for any functions f_1, f_2, f_3. It is easy to check that

$$\begin{aligned} J(h_1, h_2, h_3) &= [[r, r]]h_1h_2h_3 - h_1h_2h_3[[r, r]] && (7)\\ J(x_1, h_2, h_3) &= ([w_{12}, w_{13}] + [w_{12} + w_{13}, r_{23}])x_1h_2h_3 && (8)\\ J(x_1, x_2, h_2) &= ([r_{12}, w_{13} + w_{23}] + [w_{13}, w_{23}])x_1x_2h_3 && (9)\\ J(x_1, x_2, x_3) &= [[r, r]]x_1x_2x_3, && (10) \end{aligned}$$

where $[[\cdot, \cdot]]$ is the bracket defined by Drinfeld: for any $\rho \in \mathfrak{h} \otimes \mathfrak{h}$,

$$[[\rho, \rho]] := [\rho_{12}, \rho_{13}] + [\rho_{12}, \rho_{23}] + [\rho_{13}, \rho_{23}].$$

If $w = r$, then the Jacobi identity holds provided $[[r, r]] = 0$ (r triangular).

If $w = r + s$, where s is a symmetric invariant element of $\mathfrak{h} \otimes \mathfrak{h}$ and $[[w, w]] = 0$ (i.e. r is real-quasitriangular), then the Jacobi identity is satisfied, provided (10) is zero, i.e. the fundamental bivector field r_V on V (cf.[6]) is Poisson. We shall show that it is Poisson for almost all N, p, namely when $\mathfrak{h} = so(p, N{-}p)$ is absolutely simple. Indeed, in this case all invariant symmetric 2-tensors s are proportional to (the Killing element)

$$\widetilde{s}^{jk}_{lm} = \eta^{jk}\eta_{lm} - \delta^j_m\delta^k_l, \tag{11}$$

and all invariant elements of $\bigwedge^3 \mathfrak{h}$ are proportional to $\Omega := [[\widetilde{s}, \widetilde{s}]] = [\widetilde{s}_{12}, \widetilde{s}_{13}]$. From (11) we obtain

$$\Omega^{abc}_{jkl} = \eta^{ab}\eta_{jl}\delta^c_k + \eta^{ac}\eta_{kl}\delta^b_j + \eta^{bc}_{jk}\delta^a_l - \eta^{ab}\eta_{kl}\delta^c_j - \eta^{bc}\eta_{jl}\delta^a_k - \eta^{ac}\eta_{jk}\delta^b_l + \delta^a_k\delta^b_l\delta^c_j - \delta^a_l\delta^b_j\delta^c_k$$

which yields $\Omega^{abc}_{jkl}x^jx^kx^l = 0$. For any classical r-matrix r on $\mathfrak{h}$, $[[r, r]]$ must be proportional to Ω and therefore (10) is zero.

If $\mathfrak{h} = so(1, 3)$, all invariant symmetric 2-tensors are complex multiples of

$$\widetilde{s} = X_+ \otimes X_- + X_- \otimes X_+ + \frac{1}{2}H \otimes H \qquad (\textit{complex}\text{ tensor product}). \tag{12}$$

We use here the embedding of the complex tensor product $\mathfrak{h} \otimes_{\mathbb{C}} \mathfrak{h}$ into the real $\mathfrak{h} \otimes \mathfrak{h}$ as described in [7] (X_+, X_-, H is the standard complex basis of $so(1, 3) \cong sl(2, \mathbb{C})$ normalized as in [7]; the reader should excuse the double use of the letter H). One can check easily that

$$\widetilde{s} = \vec{M} \cdot \vec{M} - \vec{L} \cdot \vec{L}, \qquad -i\widetilde{s} = \vec{M} \cdot \vec{L} + \vec{L} \cdot \vec{M}, \tag{13}$$

where $M_i :== \varepsilon_{ijk}e_k \otimes e^j$, $L_i = e_0 \otimes e^i + e_i \otimes e^0$ $(i, j, k = 1, 2, 3)$ are standard generators of $so(1, 3)$ and therefore $\widetilde{s}$ coincides with (11). All invariant 3-vectors are complex multiples of

$$\Omega = [[\widetilde{s}, \widetilde{s}]] = X_+ \wedge H \wedge X_- \quad (\textit{complex}\text{ products; we use } \textstyle\bigwedge^3_{\mathbb{C}} \mathfrak{h} \subset \bigwedge^3 \mathfrak{h}). \tag{14}$$

Since $\Omega x_1x_2x_3 = 0$ and $(i\Omega)x_1x_2x_3 \neq 0$ (Ex. 3.3 of [6]), r_V is Poisson only if $[[r,r]]$ is (real) proportional to Ω. It means that if $r_- = i\lambda X_+ \wedge X_-$ (the only possibility of non-triangular r, up to automorphism; the notation of [7]), then $[[r,r]] = [[r_-,r_-]] = \lambda^2\Omega$, hence λ^2 must be real, i.e. λ real or imaginary (cf. [6]).

Now we turn to the question of real-quasitriangularity. From Thm. 3.3 of [8] it follows that real-quasitriangular (not triangular) r-matrices exist only in the following three cases of $so(p, N-p)$:

$$so(p,p),\ so(p,p+1) \text{ (real split cases)} \quad \text{and} \quad so(p,p+2).$$

For $so(1,1+2)$ in fact every r-matrix is real-quasitriangular (with suitable s). If it is not triangular, then, up to automorphism, $r_- = i\lambda X_+ \wedge X_-$ and $[[r,r]] = \lambda^2\Omega$, whereas $[[s,s]] = -\lambda^2\Omega$ for $s = i\lambda\widetilde{s}$, hence $[[r+s,r+s]] = [[r,r]] + [[s,s]] = 0$.

Concluding, for real-quasitriangular r such that r_V is Poisson, we have a natural Poisson structure π on G defined by (6), which generalizes π_r. This structure is not multiplicative (for $s \neq 0$). It differs from the multiplicative structure π_r only by the following brackets:

$$\{h_1,h_2\}_s = 0, \qquad \{x_1,h_2\}_s := s x_1 h_2, \qquad \{h_1,h_2\}_s = 0. \tag{15}$$

Denoting by Δ the comultiplication: $\Delta h = hh'$, $\Delta x = x + hx'$ (the primed functions refer to the *second copy* of G), we obtain

$$\{\Delta h_1, \Delta h_2\}_s = \Delta\{h_1,h_2\}_s, \qquad \{\Delta x_1, \Delta h_2\}_s = \Delta\{x_1,h_2\}_s,$$

but

$$\{\Delta x_1, \Delta x_2\}_s - \Delta\{x_1,x_2\}_s = \{\Delta x_1, \Delta x_2\}_s = (s - Ps)x_1h_2x_2', \tag{16}$$

where P is the permutation in the tensor product. It is therefore natural to look for cross-term $\{\cdot,\cdot\}_{\bowtie}$ which is nontrivial only between x and x'. With such an assumption, $(G,\pi,\pi_{\bowtie})$ will be a braided Poisson group if $\{\Delta x_1, \Delta x_2\}_s + \{\Delta x_1, \Delta x_2\}_{\bowtie} = 0$, i.e.

$$(s - Ps)x_1h_2x_2' + h_2\{x_1,x_2'\}_{\bowtie} + h_1\{x_1',x_2\}_{\bowtie} = 0. \tag{17}$$

Consider first the generic s which is proportional to (11): $s = \nu\widetilde{s}$. Since $\widetilde{s} - P\widetilde{s} = I - P$, (17) is equivalent to

$$\nu(x_1h_2x_2' - x_2h_1x_1') = h_2\{x_2',x_1\}_{\bowtie} - h_1\{x_1',x_2\}_{\bowtie}, \tag{18}$$

which is satisfied by

$$\{x_2',x_1\}_{\bowtie} = \nu x_1 x_2' \qquad \text{(more explicitly: } \{(x')^k, x^j\}_{\bowtie} = \nu x^j (x')^k\text{)}. \tag{19}$$

One has only to check that $\pi \oplus \pi + \pi_{\bowtie}$ is a Poisson bracket on $G \times G$, but this is true:

$$\begin{aligned}
J(x_1,x_2,x_3') &= \{rx_1x_2, x_3'\} + \{x_2x_3', x_1\} - \{x_3'x_1, x_2\} \\
&= 2r_{12}x_2x_3' + r_{21}x_2x_1x_3' - x_2x_1x_3' + x_3'x_2x_1 - r_{12}x_3'x_1x_2 = 0, \\
J(x_1,x_2',h_3) &= \{x_1x_2', h_3\} + \{-w_{13}x_1h_3, x_2'\} = w_{13}x_1h_3x_2' - w_{13}x_1x_2'h_3 = 0
\end{aligned}$$

(here $\{\cdot,\cdot\}$ denotes the full bracket on $G\times G$ defined by $\pi\oplus\pi+\pi_{\bowtie}$).

In the Lorentz case $\mathfrak{h}=so(1,3)$, apart from the generic case $s=\nu\widetilde{s}$, one has to consider also the case when $s=\nu i\widetilde{s}$. Using formula (13) for $i\widetilde{s}$, it is easy to see that $i\widetilde{s}-Pi\widetilde{s}=2i\widetilde{s}$ and (17) has no solutions. Thus the case of real λ in $r_-=i\lambda X_+\wedge X_-$, which corresponds to real q in the quantum case (in particular, quantum double of $SU_q(2)$), is excluded. It means that from the list of r-matrices on $so(1,3)$ in [7], only combinations of $(X_+\wedge X_- - JX_+\wedge JX_-)$ and $JH\wedge H$ fall in our scheme.

Finally, it is interesting to note that

1. the one-parameter group of automorphisms of G (dilations),
$$t(h,x):=(h,e^t x)\qquad \text{for } t\in\mathbb{R},$$
preserves π (because (6) is homogeneous in x),
2. the braiding bivector field $\pi_{\bowtie}$ described by (19) is nothing else but the antisymmetrization of the fundamental tensor field on $G\times G$ obtained by the action of the real-quasitriangular element
$$\nu e_1\otimes e_1\in\mathbb{R}\otimes\mathbb{R}\qquad (e_1 \text{ is the basic vector of } \mathbb{R}).$$

Similar property is satisfied by the *cobracket* δ on $\mathfrak{g}$, obtained by linearization of π at the group unit. It follows that $(\mathfrak{g},\delta)$ is an example of a *braided-Lie bialgebra* [9] (in the category of modules over quasitriangular $\mathbb{R}$). (G,π) will certainly be an example of a braided Poisson-Lie group, when the theory presented in [9] will be extended from Lie algebras to Lie groups.

3 The quantum case

Real-(co)quasitriangular quantum $SO(p,p)$ and $SO(p,p+1)$ are introduced in [10] and $SO(p,p+2)$ in [11]. They all can be described by relations of the form
$$Wh_1h_2=h_2h_1W,\qquad h_1h_2\eta=\eta,\ \ \eta'h_1h_2=\eta',\quad h=h^*,\tag{20}$$
where
$$\hat{W}=PW=qP^{(+)}-q^{-1}P^{(-)}+q^{1-N}P^{(0)}\tag{21}$$
is the standard R-matrix for the orthogonal series (here $P^{(+)}$, $P^{(-)}$ and $P^{(0)}$ are the spectral projections corresponding to symmetric (traceless), antisymmetric and proportional to the metric elements of $V\otimes V$) with $|q|=1$ and η' (η) is a deformed covariant (contravariant) metric. For $q=1+i\varepsilon+\ldots$ we have $W=I+i\varepsilon w+\ldots$, where w satisfies the classical Yang Baxter equation. To the skew-symmetric classical r-matrix $r=(w-w_{21})/2$ there corresponds the involutive intertwiner
$$\hat{R}:=I-2P^{(-)},\qquad R=P\hat{R}=I+i\varepsilon r+\ldots$$
(note that R can be used instead of W in (20)).

Passing to the inhomogeneous group (1), we expect that the commutation relations for g should be

$$\mathcal{R}g_1g_2 = g_2g_1\mathcal{R}, \qquad \text{where} \quad \mathcal{R} = \begin{pmatrix} R & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{22}$$

(this corresponds to r given in (3) when $a = 0$, $b = 0$). Using the form of g_1g_2 as in (3), we obtain

$$Rh_1h_2 = h_2h_1R, \qquad x_2h_1 = Rh_1x_2, \qquad h_2x_1 = Rx_1h_2, \qquad x_2x_1 = Rx_1x_2. \tag{23}$$

The two equalities in the middle are equivalent, due to the involutivity of $\hat{R}$. The last equality provides defining relations for the *quantum orthogonal vector space* [10, 11]. These relations are consistent: the corresponding algebra of polynomials has the classical size. Also the first equality gives consistent relations in this sense. It remains to check the consistency of the 'cross-relations' with other ones. From

$$R_{12}R_{13}R_{23}h_1h_2x_3 = x_3h_2h_1R_{12} = R_{23}R_{13}R_{12}h_1h_2x_3, \tag{24}$$

$$R_{12}R_{13}R_{23}h_1x_2x_3 = x_3x_2h_1 = R_{23}R_{13}R_{12}h_1x_2x_3, \tag{25}$$

it follows that R should satisfy the Yang Baxter equation, hence $q = 1$ (the triangular case). As in the Poisson case, we postulate then a modification of (23) as follows:

$$Rh_1h_2 = h_2h_1R, \qquad x_2h_1 = W'h_1x_2, \qquad x_2x_1 = Rx_1x_2, \tag{26}$$

with some matrix W'. Instead of (24)–(25), we have now

$$R_{12}W'_{13}W'_{23}h_1h_2x_3 = x_3h_2h_1R_{12} = W'_{23}W'_{13}R_{12}h_1h_2x_3,$$

$$W'_{12}W'_{13}R_{23}h_1x_2x_3 = x_3x_2h_1 = R_{23}W'_{13}W'_{12}h_1x_2x_3.$$

For the consistency of different ways of ordering, we postulate that

$$W'_{12}W'_{13}W'_{23} = W'_{23}W'_{13}W'_{12} \qquad \text{and } \hat{R} \text{ (or } P^{(-)}\text{) is a function of } \hat{W}' = PW'. \tag{27}$$

This is fulfilled if $\hat{W}'$ a scalar multiple of $\hat{W}$ (it is also possible that $\hat{W}'$ is a scalar multiple of $\hat{W}^{-1}$; this corresponds to the change $s \mapsto -s$ in the Poisson case). The scalar coefficient is not arbitrary, due to the following two conditions:

1. From the reality requirement ($h^* = h$, $x^* = x$) it follows that $x_2h_1 = W'h_1x_2$ implies $h_1x_2 = \overline{W'}x_2h_1$, hence $x_2h_1 = W'\overline{W'}x_2h_1$ and we have to assume that
$$W'\overline{W'} = I. \tag{28}$$

2. Since $x_3\eta_{12} = x_3h_1h_2\eta_{12} = W'_{13}W'_{23}h_1h_2x_3\eta_{12} = W'_{13}W'_{23}\eta_{12}x_3$, we have also the following condition of compatibility of W' with the metric:

$$W'_{13}W'_{23}\eta_{12} = \eta_{12}. \tag{29}$$

Both conditions are satisfied by $W' = W$ (another solution, $W' = -W$, has no proper classical limit). The first condition follows from

$$\overline{W(q)} = W(\overline{q}) = W(q^{-1}) = W(q)^{-1}$$

(cf. [10]; recall that $|q| = 1$). The second coincides with formula (2.21) in [12]. Thus, in the sequel we set $W' = W$.

It is easy to see that the comultiplication preserves first two relations in (26), for instance $\Delta x_2 \Delta h_1$ equals

$$(x_2+h_2x'_2)h_1h'_1 = Wh_1x_2h'_1+h_2h_1Wh'_1x'_2 = Wh_1h'_1x_2+Wh_1h_2h'_1x'_2 = W\Delta h_1\Delta x_2.$$

This will be true also for a nontrivial braiding of the type

$$x'_2x_1 = Bx_1x'_2, \tag{30}$$

which on the other hand may be used to remove the inconsistency related to the preservation of the third relation: $P^{(-)}x_1x_2 = 0$. We shall find now the condition under which $P^{(-)}\Delta x_1\Delta x_2 = 0$. The first two terms in

$$\Delta x_1\Delta x_2 = (x_1 + h_1x'_1)(x_2 + h_2x'_2) = x_1x_2 + h_1x'_1h_2x'_2 + x_1h_2x'_2 + h_1x'_1x_2$$

are annihilated by $P^{(-)}$ (second, because $P^{(-)}h_1h_2x'_1x'_2 = h_1h_2P^{(-)}x'_1x'_2 = 0$). The sum of the last two terms is equal

$$(\hat{W}h_1x_2x'_1 + h_1x'_1x_2)^{jk} = \hat{W}^{jk}_{ab}h^a_cx^bx'^c + h^j_lB^{kl}_{bc}x^bx'^c = (\hat{W}^{jk}_{ab}\delta^l_c + \delta^j_aB^{kl}_{bc})h^a_lx^bx'^c,$$

hence our condition is

$$P^{(-)}_{12}(\hat{W}_{12} + B_{23}) = 0. \tag{31}$$

If

$$P^{(-)}(\hat{W} + \sigma I) = 0 \qquad \text{for some } \sigma, \tag{32}$$

then $B = \sigma I$ is a solution of our problem and the non-trivial cross-relations are the following: $x'^jx^k = \sigma x^kx'^j$. We call (32) the *spectral condition*. Taking into account that $P^{(-)}$ is a projection and a function of $\hat{W}$, it means that $P^{(-)}$ is a spectral projection of $\hat{W}$ corresponding to a single eigenvalue. This is of course satisfied for (21), with $\sigma = q^{-1}$.

We conclude that relations (26) with $W' = W$ and braiding

$$x'^jx^k = q^{-1}x^kx'^j \tag{33}$$

define a braided quantum $ISO(p, N-p)$, which contains $SO_q(p, N-p)$.

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